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Mathematical Aspects of the Theory of Inviscid Hypersonic Flow

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Mathematical aspects of the theory of inviscid hypersonic flow

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This paper reviews some differential equations arising in the theory of inviscid hypersonic gasdynamics. The only real-gas effects that we have incorporated are simple models for chemical reactions. After describing what is known about the solution structure of these equations in unsteady one-dimensional and steady two-dimensional flow, we make some conjectures about the well-posedness and regularization of certain specific open problems which have not yet been susceptible to mathematical analysis.

1. Introduction

The purpose of this article is to recapitulate some of the simplest mathematical models for inviscid hypersonic flow with the aims of identifying (i) mathematical techniques which can provide insight into such flows, and (ii) unsolved mathematical problems associated with these models.

Although inviscid models have limited practical value, it is important to understand them as well as possible if theoretical progress is to be made with more complicated models for real gases. Our models are limiting cases of viscous or rarefied-gas models, some of which are discussed elsewhere in this issue. More important for our purposes, and sometimes vitally important in practice, is the occurrence of chemical reactions and we will see that even crude models for these reactions can be as suggestive as localized viscous modelling as far as the mathematical understanding of inviscid hypersonic flows is concerned.

We will begin our account by writing down the simplest models of ideal-gas flows, identifying the relevant parameters and commenting on the few explicit solutions which are available. Then in §2 we will describe some approximations which can be made when the flow is hypersonic and we will use these to make some conjectures about certain general properties of inviscid hypersonic flows.

The background for almost all the material presented here can be found in the famous books of Hayes & Probstein (1966) and Chernyi (1961). Many of the ideas we use were pioneered and developed in the golden age of theoretical hypersonic flow in the 1950s and 1960s. Scientific computation is now the dominant force in the analysis of hypersonic flow and the most useful attributes of modern mathematical treatments are their ability to provide test cases for numerical schemes and to give warnings where unexpected behaviour may call for extra care with discretizations.

Non-reacting flow models

We start with the equations for unsteady, one-dimensional flow of an ideal gas in the absence of viscosity, heat conduction, radiation and chemical reactions. In the

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usual dimensionless form, the eulerian equations for the density ρ, pressure p and velocity u are

$$\mathrm{d}\rho/\mathrm{d}t+\rho\,\partial u/\partial x = 0, \tag{1a}$$

$$\rho\,\mathrm{d}u/\mathrm{d}t+\partial p/\partial x = 0, \tag{1b}$$

$$\mathrm{d}e/\mathrm{d}t+p\,\mathrm{d}/\mathrm{d}t\,(1/\rho) = 0 \Rightarrow \mathrm{d}/\mathrm{d}t\,(p/\rho^{\gamma}) = 0, \tag{1c}$$

where $\mathrm{d}/\mathrm{d}t = \partial/\partial t+u\,\partial/\partial x$, $e = p/\{(\gamma-1)\,\rho\}$ and γ is the specific heat ratio, assumed constant. We will only consider flows in which there is a single shock wave adjoining an undisturbed gas in $x > x_{\mathrm{s}}(t)$, which is denoted by subscript zero; at $x = x_{\mathrm{s}}$, the Rankine–Hugoniot conditions are

$$\rho/\rho_0 = (\gamma+1)/\{(\gamma-1)+2/M^2\}, \tag{2a}$$

$$p/p_0 = \{2\gamma M^2-(\gamma-1)\}/(\gamma+1), \tag{2b}$$

$$u = 2\dot{x}_{\mathrm{s}}/(\gamma+1)\,(1-1/M^2), \tag{2c}$$

where p_0 is constant, but ρ_0 need not be, and $M^2 = \rho_0\,\dot{x}_{\mathrm{s}}^2/\gamma p_0$.

We will subsequently find it convenient to work with the lagrangian form of (1) in which x, p and ρ are considered as functions of t and the particle-path function ξ, which is such that $\partial\xi/\partial x = \rho$, $\partial\xi/\partial t = -\rho u$. Thus, assuming the gas lies in $x \geqslant 0$ at $t = 0$,

$$\xi = \int_0^x \rho_0\,\mathrm{d}x - \int_0^t \rho u\,\mathrm{d}t = \int_{x_{\mathrm{p}}(t)}^x \rho\,\mathrm{d}x,$$

where $x_{\mathrm{p}}(t)$ is the current position of the gas particle which was at the origin at $t = 0$ and can be interpreted as a piston path; ξ is the mass of gas between this particle and the point x. In these variables

$$\partial x/\partial\xi = 1/\rho, \tag{1a$'$}$$

$$\partial^2 x/\partial t^2+\partial p/\partial\xi = 0, \tag{1b$'$}$$

$$\partial/\partial t\,(p/\rho^{\gamma}) = 0, \tag{1c$'$}$$

where, in addition to (2a, b),

$$x = x_{\mathrm{s}}(t) \quad \text{at} \quad \xi = \int_0^{x_{\mathrm{s}}} \rho_0\,\mathrm{d}x, \tag{2c$'$}$$

assuming that the shock also starts at $x = 0$ at $t = 0$. We remark that since we can take p_0 to be unity, the only parameters appearing in the models are γ and the scales in ρ_0 and x_{p}, which will determine the size of M. We also note that (1) is a hyperbolic system but (1′) is not unless we work with $u = \partial x/\partial t$ instead of x.

Later, we will need to compare (1) and (2) with the equations of steady two-dimensional flow, namely

$$\mathrm{D}\rho/\mathrm{D}t+\rho\,\mathrm{div}\,\boldsymbol{u} = 0, \tag{3a}$$

$$\rho\,\mathrm{D}\boldsymbol{u}/\mathrm{D}t+\mathrm{grad}\,p = 0, \tag{3b}$$

$$\mathrm{D}/\mathrm{D}t\,(p/\rho^{\gamma}) = 0, \tag{3c}$$

where $\boldsymbol{u} = (u, v)$, $\mathrm{D}/\mathrm{D}t = u\,\partial/\partial x + v\,\partial/\partial y$, together with the Rankine–Hugoniot relations

$$\rho/\rho_0 = (\gamma+1)/\{(\gamma-1)+2/M^2\sin^2\beta\}, \tag{4a}$$

$$p/p_0 = \{2\gamma^2 M\sin^2\beta-(\gamma-1)\}/(\gamma+1), \tag{4b}$$

$$(u-U)/U = -[2/(\gamma+1)]\sin^2\beta[1-1/M^2\sin^2\beta], \tag{4c}$$

$$v/U = [2/(\gamma+1)]\sin\beta\cos\beta[1-1/M^2\sin^2\beta], \tag{4d}$$

where

$$M^2 = \rho_0 U^2/\gamma p_0 \tag{4e}$$

at a shock making an angle β with a free stream of velocity $(U, 0)$. There is now not so much advantage in working in a lagrangian frame except in a case in which the gas flow is confined to a thin layer. In the next section we will encounter circumstances under which this can happen and the relevant transformations of (3) and (4) to 'body-fitted' coordinates are given in the Appendix, together with their lagrangian counterparts.

As in the one-dimensional case, γ, U and the scales of ρ_0, and any obstacle in the flow, are the only parameters and they determine M. We could write down an equivalent of (3), (4) for axisymmetric flow with almost identical results.

Despite their apparent simplicity, there is little mathematical knowledge about the solutions of (1) or (3) except for simple geometries. For realistic values of the parameters, the most useful explicit solutions are those catalogued in gasdynamic textbooks, namely Riemann problems for shock tubes, uniformly moving pistons and flow past wedges, all of which have similarity solutions. An interesting unification of some of these flows when there is only one space variable has been given in Keller (1956). From this evidence it seems likely that, provided an entropy, or some equivalent, condition is prescribed at $x = x_{\mathrm{s}}$, the systems (1), (2) and (3), (4) are well-posed and our principal goal in the next section will be to say as much as we can about useful approximate solutions in the hypersonic limit when $M \rightarrow \infty$. Before we do this, we will catalogue some of the models for chemical reactions which have been incorporated into (1)–(4).

Reacting flows

Conceptually, the simplest models are those for which the reaction is limited to a thin region in space and incorporated as a modification to the Rankine–Hugoniot conditions to lowest order. The best-known example is the theory of detonations and deflagrations (Courant & Friedrichs 1948) in which the energy conservation statement which led to (2) is changed to

$$\left[\rho u\left(\frac{\gamma p}{(\gamma-1)\,\rho}+\tfrac{1}{2}u^2+E(p,\rho)\right)\right]_{x=x_{\mathrm{s}}-0}^{x=x_{\mathrm{s}}+0} = 0, \tag{5}$$

where E is the energy released during the reaction. This extra term not only introduces further non-uniqueness into the possible downstream conditions but, when suitable assumptions are made about E, it splits these conditions into two disjoint branches; detonations, across which p and ρ both increase and whose positions, like those of shock waves, are determined by (5), and much slower deflagrations in which p and ρ both decrease and whose positions are determined by their local structure. In fact these two phenomena can be unified by regarding a

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detonation as a shock wave followed by a deflagration. The mathematical analysis of deflagrations is especially delicate since they tend to expansion shock waves as $E \to 0$ (Ludford & Stewart 1981).

In the case of distributed chemical reactions, we need rate equations to describe how the relevant non-equilibrium variables behave behind as well as at the shock wave. For air, there is a hierarchy of such equations for the increasingly complicated molecular processes which occur as M and hence the temperature is increased. The first and simplest is the case of vibrational excitation of the molecules which is typically modelled by writing

$$e = [c_v/(c_p - c_v)](p/\rho) + c_i T_i, \tag{6}$$

where c_v, c_i and c_p are constants and the second term represents the energy in the vibrational mode. The simplest rate equation for T_i is

$$\tau\, \partial T_i/\partial t = T - T_i, \tag{6'}$$

which is to be appended to ($1a'$–c'), where the relaxation time τ is a function of p and T but usually taken as constant (Blythe 1961). The extra derivative appearing in the model necessitates an extra boundary condition behind the shock front, and, since this is typically only a few mean free paths thick, we take this as that of 'frozen' flow

$$T_i = 0 \quad \text{at} \quad x = x_s(t). \tag{2d'}$$

In this situation, there is a relatively wide region downstream of the shock in which the vibrational modes are excited.

We note that the model ($6, 6'$) can be related to (5) in two simple cases. First, when $\tau = 0$, the flow is in equilibrium everywhere downstream of the shock and is therefore modelled by (1) or ($1'$) with γ replaced by $\gamma_e = c_p/(c_v + c_i)$ and the shock relations (2) or ($2'$) with γ replaced by γ_f. A simple calculation shows that these shock relations are the same as those with γ replaced by γ_e as long as $E = [c_i/(c_p - c_v)]\, p/\rho$. Secondly, when $\tau = \infty$, we can work with γ_f in both the field equations and the shock conditions.

At higher Mach numbers and temperatures, the molecules may dissociate into individual atoms and the rate equation is now conventionally written down for the fraction of dissociated molecules (Freeman 1958; Sundaram 1968). The relation between these models and ($6, 6'$) is discussed in Spence (1961). Finally, at yet higher temperatures, ionization and radiation can become important (Vincenti & Kruger 1965), but we will not discuss these here.

Without incorporating extra information such as the Chapman–Jouguet relation for (5), or letting τ tend to zero or infinity in ($6'$), there are no explicit solutions for the chemically reacting models described above. However, approximations can be made in various limiting cases, especially for weak shock waves (Clarke 1960; Moore & Gibson 1960) and we will see in the next section that similar progress can also be made in the hypersonic limit.

2. Approximate solutions for hypersonic flow

Small disturbance theory

The most famous simplification in inviscid hypersonic flow theory relates (1, 2) to (3, 4) when the latter are applied to flow past a body which is uniformly thin, i.e. of the form $y = \delta f(x)$ where $\delta \ll 1$ and $|f'| \lesssim O(1)$ throughout. For $1 \gg 1/M \gg \delta$, we may still use supersonic small disturbance theory; however, when $M\delta \gtrsim O(1)$, so that the leading Mach wave in supersonic small disturbance theory has an inclination comparable to that of the body, we can no longer treat this Mach wave as a weak shock and a new scaling becomes necessary. The motivation for these scalings is given in (Ockendon & Taylor 1983), repeated here for convenience. We only consider the case of a uniform free stream, denoted by a subscript zero.

Supersonic small disturbance theory

$$u = U(1+\delta\bar{u}),$$
$$v = \delta U\bar{v},$$
$$p = p_0+\rho_0 U^2\delta\bar{p},$$
$$\rho = \rho_0(1+\delta\bar{\rho}),$$
$$x = \bar{x}, \quad y = \bar{y},$$

Hypersonic small disturbance theory

$$u = U(1+\delta^2 u^*),$$
$$v = \delta U v^*,$$
$$p = \rho_0 U^2\delta^2 p^*,$$
$$\rho = \rho_0\rho^*,$$
$$x = x^*, \quad y = \delta y^*.$$

To lowest order

$$\left(\frac{\partial\bar{u}}{\partial\bar{x}}+\frac{\partial\bar{v}}{\partial\bar{y}}\right)+\frac{\partial\bar{\rho}}{\partial\bar{x}} = 0, \qquad (7a)$$

$$\frac{\partial(\rho^* v^*)}{\partial y^*}+\frac{\partial\rho^*}{\partial x^*} = 0, \qquad (7a')$$

$$\frac{\partial\bar{u}}{\partial\bar{x}} = -\frac{\partial\bar{p}}{\partial\bar{x}}, \qquad (7b)$$

$$\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*} = -\frac{1}{\rho^*}\frac{\partial p^*}{\partial x^*}, \qquad (7b')$$

$$\frac{\partial\bar{v}}{\partial\bar{x}} = -\frac{\partial\bar{p}}{\partial\bar{y}}, \qquad (7c)$$

$$\frac{\partial v^*}{\partial x^*}+v^*\frac{\partial v^*}{\partial y^*} = -\frac{1}{\rho^*}\frac{\partial p^*}{\partial y^*}, \qquad (7c')$$

$$\frac{\partial\bar{p}}{\partial\bar{x}} = \frac{1}{M^2}\frac{\partial\bar{\rho}}{\partial\bar{x}}, \qquad (7d)$$

$$\left(\frac{\partial}{\partial x^*}+v^*\frac{\partial}{\partial y^*}\right)\left(\frac{p^*}{\rho^{*\gamma}}\right) = 0, \qquad (7d')$$

together with the boundary conditions

$$\bar{v} = f'(\bar{x}) \quad \text{on} \quad \bar{y} = 0, \qquad (7e) \qquad\qquad v^* = f'(x^*) \quad \text{on} \quad y^* = f(x^*). \qquad (7e')$$

In the hypersonic case we must also append the shock conditions in the form

$$\rho^* = (\gamma+1)\{(\gamma-1)+2/M^2\beta^2\}^{-1}, \qquad (8a)$$

$$p^* = [2/(\gamma+1)]\, Y_{\mathrm{s}}'^2(1-(\gamma-1)/2\gamma M^2\beta^2), \qquad (8b)$$

$$u^* = -[2/(\gamma+1)]\, Y_{\mathrm{s}}'^2(1-1/M^2\beta^2), \qquad (8c)$$

$$v^* = [2/(\gamma+1)]\, Y_{\mathrm{s}}'(1-1/M^2\beta^2), \qquad (8d)$$

on the shock $y = \delta Y_{\mathrm{s}}(x)$, where $\beta = \delta Y_{\mathrm{s}}'$.

By proceeding to second order in the hypersonic theory, and expanding for large values of u^*, y^* and p^*, we can retrieve the supersonic theory but more important is the fact that ($7b'$) uncouples from ($7a'$, $7c'$, $7d'$) so that ($7'$) is formally equivalent to the unsteady one-dimensional model (1), (2) in the hypersonic limit; the body boundary condition ($7e'$) corresponds to the motion of a piston following a path $x_{\mathrm{p}} = f(t)$. A similar result relates steady hypersonic slender body flow to unsteady two-dimensional piston motion.

We emphasize that the body shape must be uniformly small for the above remarks to be valid and that any blunting of the leading edge or tip becomes a more and more important global feature of the flow as the Mach number is increased. This is reflected in the fact that the shock conditions are all-important even in slightly perturbed hypersonic flows and hence any blunting, which produces a locally much stronger shock wave than is present further downstream, engenders a layer of gas close to the body in which the scalings for the velocity, density and temperature are different from (7′); these variables could only change by order unity across the layer if the free stream was merely supersonic. The structure of this 'entropy layer' has been described in Yakura (1962) and Rusanov (1976), and we will not discuss it further here except to note that the precise form of the blunting not only crucially affects the behaviour of the entropy layer, but it can alter the entire flow pattern, including the shock position, in the case when the blunting radius is of $O(\delta)$. This fact led to the proposal (Chernyi 1960) that blunted thin or slender bodies could be modelled by adding an empirically-determined localized drag force to the momentum equations; in the unsteady analogy, this drag corresponds to a local energy release at the start of the piston motion. We will return to this interesting idea again when we re-examine the piston problem, (1), (2) in the hypersonic limit.

It is interesting to note that the hypersonic small disturbance analogies carry over to some of the chemically reacting flows mentioned in the introduction. If the reaction is modelled by the localized shock condition (5), the analogy will still apply as long as E scales with p/ρ although the Rankine–Hugoniot conditions will now be much more complicated in general. In Barenblatt (1979), it was noted that in the case when the energy loss E is simply $E_0\, p/\rho$, the energy condition in the hypersonic limit reduces to that in the nonreacting case except that in the Rankine–Hugoniot conditions γ is replaced by Γ where

$$\Gamma/(\Gamma-1) = \gamma/(\gamma-1)-E_0, \tag{9}$$

thus γ is effectively reduced in value when E_0 is positive and tends to unity as the energy loss increases. Roughly similar remarks apply to the case of distributed reactions say, with e having the form (6). Since e, T and T_{i} all scale with p/ρ, the hypersonic small disturbance form of the energy equation is

$$(\partial/\partial x^* + v^*\,\partial/\partial y^*)\,(c_{\mathrm{v}}\, T^* + c_{\mathrm{i}}\, T_{\mathrm{i}}^*) + p^*(\partial/\partial x^* + v^*\,\partial/\partial y^*)\,(1/\rho^*) = 0$$

where the temperature changes are scaled with $U^2\delta^2$. Moreover the rate equation (6′) becomes

$$\tau(\partial/\partial x^* + v^*\,\partial/\partial y^*)\,T^* = T^* - T_{\mathrm{i}}^*,$$

and again $T_{\mathrm{i}}^* = 0$ immediately behind the shock. Other more complicated rate equations may also admit the hypersonic small disturbance analogy as long as they scale appropriately. Indeed, setting aside the difficulties associated with the entropy layer, the prevalence of the analogy indicates that we need as clear an understanding as possible of unsteady piston motion in the hypersonic limit.

Piston motions and similarity solutions

The non-reacting one-dimensional hypersonic piston problem is, in lagrangian variables, (1′) with

$$\rho = [(\gamma+1)/(\gamma-1)]\,\rho_0, \quad p = 2\rho_0\,\dot{x}_{\mathrm{s}}^2/(\gamma+1), \quad x = x_{\mathrm{s}}(t) \tag{9′}$$

at
$$\xi = \rho_0\, x_{\mathrm{s}}(t),$$

where for the moment, we restrict attention to the simplest case $\rho_0 = \text{const}$. It is fortunate that in the case of a 'power-law' piston motion

$$x_{\mathrm{p}} = ct^{\alpha} \quad \text{at} \quad \xi = 0, \quad c = \text{const.},$$

these simplified shock conditions admit a group symmetry of the field equations (1′) and boundary conditions (2′). Indeed, writing

$$x = t^{\alpha} X(\eta), \quad p = \rho_0 t^{2(\alpha-1)} P(\eta), \quad \rho = \rho_0 R(\eta),$$

where $\eta = \xi/\rho_0 t^{\alpha}$, we can seek a solution in which

$$x_s = bt^{\alpha}, \quad b = \text{const.}$$

and

$$\eta^2 X'' + (\alpha^{-1} - 1)(\eta X' - X) + \alpha^{-2} P' = 0, \tag{10a}$$

$$RX' = 1, \tag{10b}$$

$$(PR^{-\gamma}\eta^{2/\alpha-2})' = 0, \tag{10c}$$

where $' = \mathrm{d}/\mathrm{d}\eta$. The shock and piston boundary conditions are

$$X(b) = b, \quad P(b) = 2b^2/(\gamma+1), \quad R(b) = (\gamma+1)/(\gamma-1) \tag{10a'}$$

and

$$X(0) = c \tag{10b'}$$

respectively. We can easily rewrite the system (10) as a single second-order equation for X which has a further scale invariance so that it can be written as a first-order equation for $\mathrm{d}W/\mathrm{d}V$ where

$$s = \ln \eta, \quad V(s) = \mathrm{e}^{-ps} X(\eta), \quad W(s) = \mathrm{d}V/\mathrm{d}s \quad \text{and} \quad p = 1 - 2/\{\alpha(\gamma+1)\}.$$

The details are given in Louie (1991) but the principal feature of the resulting phase-plane analysis is that for values of α near $\frac{2}{3}$, the piston path, namely the origin, is only accessible from the shock on a trajectory on which V is positive if $\alpha > \frac{2}{3}$ (this restriction on α is usually obtained from a phase-plane analysis in eulerian variables, for example Grigorian (1958) and Lees & Kubota (1957)). Thus it seems that the group symmetries can only be exploited for these values of α.

The physical significance of this critical value of α becomes apparent when we consider the energy supplied to the gas by the piston over the time interval $(0, T)$. Either by calculating

$$\int_0^T p(0,t)\,\dot{x}_{\mathrm{p}}\,\mathrm{d}t \quad \text{or} \quad \int_0^{x_s(T)} \left(\frac{p}{(\gamma-1)\rho} + \tfrac{1}{2}u^2\right)\mathrm{d}\xi,$$

we find this is proportional to $T^{3\alpha-2}$ and hence only finite for $\alpha > \frac{2}{3}$. The case in which $x_s \propto t^{\frac{2}{3}}$ is the famous 'blast-wave' solution of Taylor (1950) and Sedov (1959), which models an instantaneous finite energy release, with no associated piston motion. The implications of the restriction $\alpha \geqslant \frac{2}{3}$ for steady two-dimensional flow have been discussed frequently. For flow past thin power-law bodies with exponent $\leqslant \frac{2}{3}$, it has been suggested (Cheng & Pallone 1956; Freeman *et al.* 1964; Hornung 1967) that the shock is still analogous to a blast wave, i.e. eventually has a shape $Y_s \propto x^{\frac{2}{3}}$, but with a coefficient determined empirically by the drag exerted on the nose.

The assumption of a strong enough shock that the terms in M^{-2} in (8) may be neglected permit many other similarity solutions to be considered, say for gas expanding into a vacuum (Grundy & McLaughlin 1977), implosions (Stanyukovich 1960) and, of more relevance to us, shocks propagating into inhomogeneous atmospheres in which ρ_0 is proportional to some power α' of the distance from the initial shock position (Raizer 1964). This latter case has some relevance to the discussion above because if α' is negative, so that the density at the origin is infinite, then similarity solutions can be found if $\alpha > 2/(3+\alpha')$.

Because rate equations introduce new timescales into the model, it is difficult to find explicit solutions to reacting piston flows. However, similarity solutions with curious properties have been found for the special model introduced in Logan & Woerner (1989). One notable result in an extreme case is that of Barenblatt (1979). By introducing a strong energy loss at the shock, so that $\Gamma \to 1$ in (9), the difficulty associated with $\alpha \leqslant \frac{2}{3}$ can be formally removed and a similarity solution written down for all $\alpha \geqslant 0$.

The breakdown of the similarity solution (10) at the 'blast wave' value of $\alpha = \frac{2}{3}$ poses several interesting mathematical questions and in particular whether any solution at all exists for $\alpha \leqslant \frac{2}{3}$ and whether any piston motion exists which produces a shock wave in which $x_s \propto t^{\beta}, \beta \leqslant \frac{2}{3}$. (Professor W. Chester has remarked that this kind of shock motion may be possible if an initial energy release is followed by a piston withdrawal. This would be in accord with the phase-plane discussion following (10).) Similar breakdowns in other kinds of similarity solutions for one-dimensional hypersonic flow have been listed in Barenblatt (1979), which also suggests remedies in the form of various kinds of regularization of the initial data, but which still permit the solution to tend to a similarity solution for large times. This philosophy of regularization of similarity solutions has a long history in applied mechanics (see, for example, Moffatt & Duffy 1980) and will play an important role in our subsequent discussion of piston motion. Before pursuing this, we introduce one other approximation into our model (1′) which will not only enable us to make conjectures about similarity solutions to the piston problem, both with and without chemical reactions, but will also shed light on some blunt body flows.

Newtonian theory

If we make the physically questionable assumption that $0 < (\gamma-1)/(\gamma+1) = \epsilon \ll 1$ and attempt an expansion of the solution of the hypersonic models (1′), (2′) in powers of ϵ, the large constant density ratio across the shock immediately suggests that the flow behind this shock is confined to a thin 'shock layer'. In such a situation the formal lowest order solution is

$$x_s = x_p, \tag{11a}$$

$$x = x_p, \tag{11b}$$

$$p = -\xi \ddot{x}_p + x\ddot{x}_p + \dot{x}_p^2, \tag{11c}$$

$$\rho = p/G(\xi), \tag{11d}$$

where

$$G(x_p(t)) = \dot{x}_p^2(t). \tag{11e}$$

A surprisingly similar result holds in the case of two-dimensional steady flow past an arbitrary (not necessarily thin) body, as long as body fitted coordinates are used as described in (A 1)–(A 4). (The relation between this configuration and that proposed by Newton (1729) is described in detail in Hayes & Probstein (1966) and Chernyi (1961).) In these coordinates, the result is

$$X_s = X_b(x), \tag{12a}$$

$$y = 0, \tag{12b}$$

$$p = \sin^2 \phi(x) + \kappa(x) \int_{X_b(x)}^{\psi} \cos\phi(X_b^{-1}(\psi))\, d\psi, \tag{12c}$$

$$\rho = p/\sin^2\phi(X_b^{-1}(\psi)), \tag{12d}$$

$$u = \cos\phi(X_b^{-1}(\psi)), \tag{12e}$$

where $X_b(x)$ is the body thickness, the body has slope ϕ and curvature κ, and $X_b^{-1}(\psi)$ is the x coordinate of the intersection of the streamline through (x, ψ) with the lowest order shock position. In fact, the approximation can be applied to general three-dimensional flows in which particles within the shock layer travel along geodesics; this introduces the interesting new possibility that particles entering the shock layer at two distinct points can both arrive at the same point on a suitably concave body at the same time. Such 'shock line' formation is discussed in Hayes & Probstein (1966).

The principal limitation on the applicability of (11) and (12) is apparent when we attempt to compute the shock layer thickness to lowest order. From (11) this gives

$$x_s - x_p = \epsilon \int_0^{x_p(t)} \frac{G(\xi)\,d\xi}{\{x_p \ddot{x}_p + \dot{x}_p^2 - \xi \ddot{x}_p\}} \tag{13}$$

and, from (12),

$$X_s - X_b = \epsilon \int_0^{X_b(x)} \sin^2\phi(X_b^{-1}(\psi'))\,d\psi' \bigg/ \left\{\sin^2\phi(x) + \kappa(x)\int_{X_b(x)}^{\psi'} \cos\phi(X_b^{-1}(\psi))\,d\psi\right\} \tag{13$'$}$$

as long as the integrals on the right-hand side exist. This can fail to be the case when the piston path (or the body) is either sufficiently abrupt at $t = 0$ ($x = 0$) that G (or $\sin^2\phi$) has a non-integrable singularity there or when $x_p \ddot{x}_p + \dot{x}_p^2$, or

$$\sin^2\phi(x) - \kappa(x)\int_0^{X_b} \cos\phi(X_b^{-1}(\psi))\,d\psi,$$

vanishes at some instant (position). When $x_p \propto t^\alpha$ (or $X_b \propto x^\alpha$), the former situation corresponds to $\alpha \leqslant \frac{2}{3}$ ($\alpha \leqslant \frac{1}{2}$), which is especially interesting in view of the discussion above. In either case the lowest order density vanishes so rapidly as the piston (body) is approached that the boundary condition there can seemingly only be satisfied if the shock is much further away than $O(\epsilon)$.

This eventuality immediately suggests that shock layers in our fictitious newtonian gas can 'fly off' or separate from the piston or body responsible for their existence. In the case of steady flow, the local details of such separations have been discussed in Freeman (1960) and Ockendon (1966, part II is in error) but the important new global feature is the possibility (Lighthill 1957; Hayes & Probstein 1966) that, downstream of the separation point, the shock layer becomes a 'free layer' separated from the piston or body by a relatively large region of low pressure, low density gas. The possibility of such a configuration can be envisaged by considering the Prandtl–Meyer expansion of a gas whose Mach number is $O(\epsilon^{-\frac{1}{2}})$, as is the case in a newtonian shock layer. Then it can be seen that if the flow is past a corner of angle much larger than $O(\surd\epsilon)$, almost all the gas is contained in a wedge of angle $O(\surd\epsilon)$, as in figure 1, with the pressure and density of the gas downstream of this wedge being $O(e^{-1/\surd\epsilon})$. A similar result applies to the one-dimensional flow which occurs when a piston moving into a gas at constant velocity suddenly has its velocity reduced to a new constant value. It is interesting to note that in both these cases, no matter how large the corner angle nor how large the piston velocity ratio, the resulting expansion is never enough to produce a vacuum in the newtonian limit; the gas can merely expand to exponentially small pressures and densities.

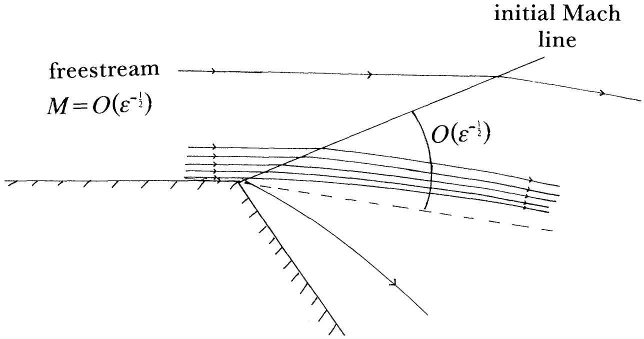

Figure 1. Prandtl–Meyer expansion of a newtonian shock layer.

The lowest order shape of a free layer is given by equating (11*c*) or (12*c*) to zero at its base, which is effectively where ξ or ψ is zero; this gives

$$x_s = at^{\frac{1}{2}} \quad \text{or} \quad X_s = a'x^{\frac{1}{2}},$$

for some constants a, a' whose determination will be considered later.

The more detailed structure of free layers, their regions of validity, and their relation to blast waves will be discussed in the final section but, in view of the simplicity of the lowest order solutions (11), (12), we first consider the effect on these solutions of some of the chemical reactions listed in §1.

Chemically reacting newtonian flows

Models for reacting hypersonic flows are generally so complicated as to necessitate either a numerical treatment (Capiaux & Washington 1963; Sedney *et al.* 1964; Hall *et al.* 1962; many more recent references can be found in Anderson 1989) or a linearization in a situation where disturbances are uniformly small (Clarke 1960; Moore & Gibson 1960). However, the newtonian non-reacting solution (11) is so simple that it permits the incorporation of models such as (6, 6′) for vibrational relaxation. We now define a small positive number ϵ' such that $\gamma_e = 1+2\epsilon'$, $\gamma_f = 1+2\epsilon'(1+\lambda)$; λ is positive since c_v and c_i are both positive and γ_e, the specific heat ratio for a gas in which $T = T_i$, corresponds to a specific heat $c_v + c_i > c_v$. Thus, when we eliminate T_i between the energy equation (1*c*′) and the rate equation (6′), we obtain

$$(\partial^2/\partial t^2 + \nu\,\partial/\partial t)\,p/\rho = 0, \quad \nu = (1+\lambda)/\tau \tag{14}$$

to lowest order, with the extra shock condition

$$\partial/\partial t(p/\rho) + \lambda\nu\dot{x}_s^2 = 0 \quad \text{on} \quad x = \dot{x}_s(t). \tag{14′}$$

The lowest order solution is now

$$x_s = x_p, \quad x = x_p, \quad p/\rho = G(\xi) + H(\xi)\,e^{-\nu t},$$
$$G(x_p) = \dot{x}_p^2, \quad H(x_p) = \lambda\dot{x}_p^2\,e^{\nu t}.$$

When we write $p = (1-\epsilon'(\lambda+1))\,\dot{x}_s^2$ instead of (9*b*′) and proceed to the second order terms in ϵ' in this newtonian expansion, we find the interesting result that the pressure on a piston moving with constant velocity c is

$$p = c^2[1+\epsilon'[1+\{1-\nu t\}\,\lambda\,e^{-\nu t}]] \tag{15}$$

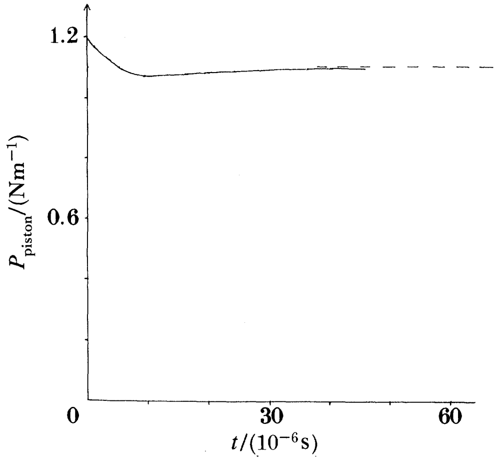

Figure 2. Pressure on constant velocity piston in vibrationally relaxing gas, showing non-monotonic approach to equilibrium value. Data: $\rho_0 = 1$ kg m^{-3}, $c = 1$ m s^{-1}, $\tau = 10^{-5}$ s, $\lambda = 1$, $\epsilon = 0.1$. ($\gamma_e = 1.2$, $\gamma_f = 1.4$.)

and therefore decreases from its initially frozen value through a minimum before finally increasing to its equilibrium value as $t \to \infty$. For typical values of the parameters in (15) this non-monotonic region is very small (see figure 2). By invoking the hypersonic small disturbance analogy, this non-monotonicity may be compared with that obtained numerically by Sedney & Gerber (1963) and Lee (1965) in the case of flow past a slender wedge. For more complicated rate equations, this non-monotonic behaviour appears to be absent (see Capiaux & Washington 1963; Stalker 1989).

Modifications to (3*c*) analogous to (14) have also been obtained for blunt body flows, and also in the presence of more complicated reactions, in Freeman (1958) and Hornung (1972).

We note with disappointment that dissipative processes such as (6, 6′) have not alleviated the singularity when $x_p = ct^{\alpha}, \alpha \leqslant \frac{2}{3}$. The integrand in the formula (13) for the shock layer thickness is smaller when relaxation is present, but it is still nonintegrable at $\xi = 0$. Hence these reactions are incapable of resolving the singularity. A similar remark applies to the smoothing effect of incorporating a finite Mach number in the shock condition, the principal consequence of which is to change the shock conditions for the density from $\rho = \epsilon$ to $\rho = \epsilon(1+\delta/\dot{x}_s^2)$, where $\delta = 2\gamma p_0/(\gamma+1)\,\rho_0$ and, since $|\dot{x}_s| \to \infty$ as $t \to 0$, there is no singular behaviour as $\delta \to 0$. We recall that even the presence of an ambient gas with a power-law density distribution $\rho_0 \propto x^{\alpha'}$ only changes the shock conditions to $\rho \propto x_s^{\alpha'}$ on $\xi \propto x_s^{\alpha'+1}$. This means that when $x_s \propto t^{\alpha}$, $G(\xi) \sim \xi^{(2-2/\alpha)/(1+\alpha')}$ as $\xi \to 0$ and the similarity solution ceases to exist when $\alpha \leqslant 2/(3+\alpha')$. The critical value of α is thus merely reduced when $\alpha' > 0$. The only mechanism we have yet uncovered which removes the criticality altogether is that of infinite energy loss at the shock as in Barenblatt (1979).

It is with these thoughts in mind that a scenario for such piston flows with $\alpha \leqslant \frac{2}{3}$ is proposed in the next section.

Separated newtonian flows

The preceding sections have revealed the need for a clearer understanding of the most interesting phenomenon uncovered by the newtonian approximation, namely that of shock-layer separation. We will confine our remarks here to unsteady one-dimensional motion, but some of the ideas will be applicable to steady flow in more dimensions.

We expect both on physical grounds and from the equality of the critical values of α in (10) and (11) that, in the limit as $\gamma \to 1$, the non-existence of similarity solutions to power-law piston problems is related to the separation of newtonian shock layers. What we will suggest in this section is that newtonian power-law piston problems with $\alpha \leqslant \frac{2}{3}$ are only well-posed when some regularization or smoothing of the piston motion is imposed near $t = 0$ and that this regularization determines certain parameters describing the subsequent flow without affecting its gross features. This kind of description is in line with our earlier references (Chernyi 1960; Barenblatt 1979; Moffatt & Duffy 1980).

Our argument is based on the naive lowest-order approximation to (11), namely

$$\partial^2 x/\partial t^2 + \partial p/\partial \xi = 0, \quad p/\rho = G(\xi), \quad \partial x/\partial \xi = \epsilon/\rho, \tag{16}$$

where $G(x_s) = \dot{x}_s^2$. These equations give

$$(\partial x/\partial \xi)^2\, \partial^2 x/\partial t^2 = \epsilon\{G(\xi)\, \partial^2 x/\partial \xi^2 - G'(\xi)\, \partial x/\partial \xi\} \tag{17}$$

together with the shock conditions

$$x = x_s(t), \partial x/\partial \xi = \epsilon \quad \text{on} \quad \xi = x_s(t). \tag{17$'$}$$

The neglect of the higher-order terms in the asymptotic expansion in ϵ will be justifiable *a posteriori*. We note that the exact version of (16) was written down and solved in several special cases in Keller (1956), but here we discuss the question of the well-posedness of (16) for small ϵ when the piston boundary condition

$$x = x_p(t) = ct^\alpha \quad \text{on} \quad \xi = 0$$

is also imposed.

The behaviour of the similarity solutions to (17), (17′) is even easier to understand than that of (10). In neither case can the ordinary differential equation for $X(\eta)$ have bounded solutions as $\eta \to 0$ when $\alpha \leqslant \frac{2}{3}$. Not surprisingly, the retention of higher order terms in ϵ in (16) will not retrieve the situation nor, as mentioned earlier, will either the retention of finite Mach number effects or the introduction of a power-law ambient density.

We have remarked earlier that (Hayes & Probstein 1966; Lighthill 1957) have suggested that in two- or three-dimensional steady flow, newtonian shock layers will separate from bodies and form free layers at points where the lowest-order newtonian pressure (12c) vanishes. We now consider applying this idea for sufficiently small times, including $t = 0$, for piston paths with $\alpha \leqslant \frac{2}{3}$. This would suggest that, from (11c), $x_s \sim at^{\frac{1}{2}}$, at least for $\frac{2}{3} \geqslant \alpha > \frac{1}{2}$, because we must satisfy $x_s > x_p$ for small enough t; however, this leaves open the problem of determining the coefficient a. We thus seek a matched asymptotic expansion model in which the solution of (17) is such that $x \sim x_s = at^{\frac{1}{2}}$ (i.e. a 'free layer' solution) for $\xi = O(1)$ but x varies from x_p to x_s for sufficiently small values of ξ. Now the right-hand side of (17), which repre-

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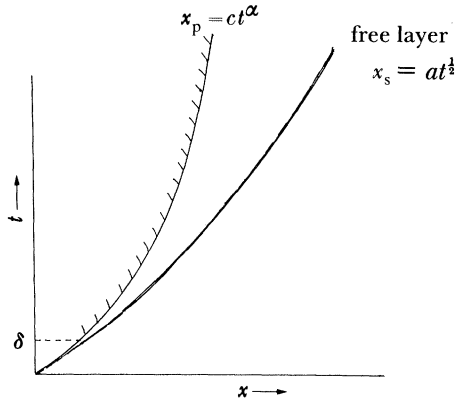

Figure 3. Regularization of piston path, showing conventional shock layer for $t < \delta$.

sents the particle path curvature, is only comparable with the left-hand side when $\xi' = \epsilon^{-\frac{1}{2}}\xi = O(1)$ and hence, for $\frac{2}{3} \geqslant \alpha > \frac{1}{2}$, we would like to be able to solve

$$(\partial x/\partial \xi')^2\, \partial^2 x/\partial t^2 = \tfrac{1}{4}a^4(\xi'^{-2}\,\partial^2 x/\partial \xi'^2 + 2\xi'^{-3}\,\partial x/\partial \xi'),$$

with $$x \sim at^{\frac{1}{2}} \quad \text{as} \quad \xi' \rightarrow \infty \qquad (18a)$$

and $$x = ct^\alpha \quad \text{on} \quad \xi' = 0. \qquad (18b)$$

However, it is simple to observe by balancing powers of ξ' that, as in the case of the similarity solution, this equation seems to have no solutions in which x is finite as $\xi' \rightarrow 0$ and nor would it if any of the remedies suggested in the previous paragraph were attempted.

In view of all these remarks, we resort to a regularization of the piston motion which introduces a conventional shock layer for $t < \delta$. We write the piston path as a continuously differentiable function of the form

$$x_p = \begin{cases} x_{p0}(t), & 0 < t < \delta, \\ ct^\alpha, & \delta < t, \end{cases} \qquad (19)$$

where $x_{p0} = O(t^{\alpha_0})$, $\alpha_0 > \frac{2}{3}$ as $t \rightarrow 0$ (see figure 3). Since $x_s \sim x_{p0}$ for $t < \delta$, this means that the small ξ behaviour is now changed so as to satisfy $G(x_{p0}) = \dot{x}_{p0}^2$ for $t < \delta$; hence $G(\xi) = O(\xi^{2-2/\alpha_0})$ as $\xi \rightarrow 0$. It is now a simple matter to join a unique free layer solution for $t > \delta$ smoothly to a shock layer solution for $t < \delta$. This involves solving $\ddot{x}_s^2 + \dot{x}_s^2 = 0$ for $t > \delta$ with x_s and $\dot{x}_s$ prescribed at $t = \delta$ and gives $x_s = at^{\frac{1}{2}}$ for some value of a dependent on x_{p0}. This value of a will have to increase by an order of magnitude as $\alpha \rightarrow \frac{1}{2}$ so that x_s should continue to exceed x_p as this limit is approached.

The precise details of the way in which the shock and free layers match near $t = \delta$ can be given in the case $x_{p0} \propto t$. Then G is approximately constant in the transition region and (17) becomes an equation which is identical to one which was analysed in Ockendon (1966) in connection with the separation of a steady two-dimensional newtonian shock layer from a wedge at a point where it faired smoothly into a cylinder. It can be shown that the shock position automatically becomes parabolic at the end of the transition region. We will return to the relation between newtonian piston motions and two-dimensional blunt body flows in the conclusion but unfortunately this seems to be the only configuration in which the genesis of the free layer can be analysed explicitly.

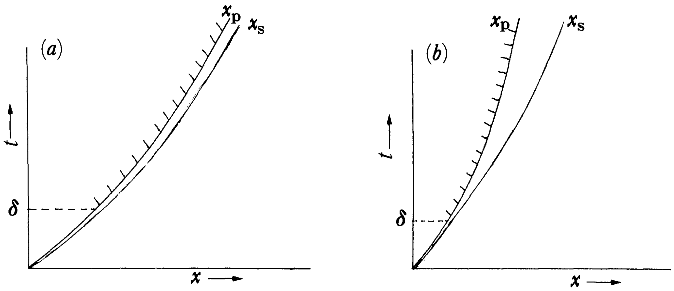

Figure 4. (*a*) No free layer downstream of the regularization region. (*b*) Free layer is joined smoothly to the shock layer at $t = \delta$.

We can now give a scenario for flows produced by piston motions for arbitrary α. We must initiate the motion with $x_p = x_{p0}$ as in (19). When we compute the smoothly joining layer $x_s(t)$ for $t > \delta$, then either (i) the subsequent piston motion, which may be $x_p \propto t^\alpha, \alpha \leqslant \frac{2}{3}$, is greater than x_s for $t > \delta$; in this case no separation occurs and the newtonian shock layer continues to be an acceptable first approximation for $t > \delta$ figure 4*a*† or (ii) the subsequent piston motion is less than x_s. Then there is a downstream free layer described by (18*a*), but where the crucially important behaviour of G for small ξ is determined by x_{p0} (figure 4*b*).

Within this latter case, there are still two possibilities, even when we exclude smooth motions in which $\alpha_0 > 1$. Either $\alpha_0 < 1$, in which case $G(\xi) \sim \xi^{2-2/\alpha_0}$ as $\xi \rightarrow 0$ and G is such that we can solve (17) for small ϵ by matched asymptotic expansions involving algebraic powers of ϵ, with $\xi\epsilon^{-\alpha_0/2(1-\alpha_0)}$ being the relevant inner variable, which is of order one in the region between x_p and x_s. Within this separated region, the pressure and density are algebraically small in ϵ. This can be thought of as a 'blunted' piston motion which introduces an 'entropy layer' between the free layer and the piston. Alternatively, when $\alpha_0 = 1$ so that $G \sim O(1)$ as $\xi \rightarrow 0$, (17) effectively becomes the linearizable equation mentioned earlier,

$$(\partial x/\partial \xi)^2\, \partial^2 x/\partial t^2 = \epsilon/4(\partial^2 x/\partial \xi^2) \tag{19}$$

as $\xi \rightarrow 0$. No algebraic scaling is now available to balance the two sides of this equation for small ξ but a heuristic estimate of the way in which x can vary from $at^{\frac{1}{2}}$ for relatively large values of ξ to x_p at $\xi = 0$ can be obtained by writing $x \sim at^{\frac{1}{2}} + \epsilon\tilde{x}$ where, to lowest order,

$$\partial^2\tilde{x}/\partial\xi^2 + at^{-\frac{3}{2}}(\partial\tilde{x}/\partial\xi)^2 = 0.$$

The relevant solution of this ordinary differential equation is

$$\tilde{x} = (t^{\frac{3}{2}}/a)\ln(\xi/at^{\frac{1}{2}}),$$

which suggests that it is only when ξ is exponentially small in ϵ that $|\tilde{x}|$ can become large. Such exponentially small terms have already been foreshadowed by the newtonian limit of the Prandtl–Meyer expansion referred to earlier, and can also be compared with newtonian source and vortex flows (Ockendon 1965).

We conclude by mentioning that piston motions which start smoothly enough for newtonian theory to be easily applicable at $t = 0$, but which are subsequently slowed

† If, in this case, the piston velocity was discontinuous at $t = \delta$, a localized force would be exerted on the piston at $t = \delta$ as described in Hayes & Probstein (1966).

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abruptly, can be handled similarly because there is no need for δ to be a small parameter in any of the above analysis.

3. Conclusion

We have presented as complete an account as we have been able of the principal new phenomena encountered in the simplest theories of inviscid hypersonic flow. We have concentrated on the case of one-dimensional piston motion for the sake of mathematical convenience, but many of our results can, with minimal complication, be applied to two-dimensional or axially symmetric flows past blunt obstacles. However, in these cases we would have to examine the details of the flow near the nose much more carefully because of the subsonic region near the stagnation point. This can cause even more difficulties with the local divergence of the newtonian approximation than was the case with our one-dimensional piston motions. Nonetheless, if we only consider steady flow past a pointed body, we can set this difficulty aside and the shock layer formulation of (A 1)–(A 4) can be rewritten as

$$\tilde{F}(\psi)\left(\frac{\partial Y}{\partial \psi}\right)^2\left(\kappa(x)-\epsilon\frac{\partial^2 Y}{\partial x^2}\right)=\tilde{G}'(\psi)\frac{\partial Y}{\partial \psi}-\tilde{G}(\psi)\frac{\partial^2 Y}{\partial \psi^2} \tag{20}$$

to lowest order, where $\tilde{F}$, $\tilde{G}$ and κ depend only on the geometry of the body. In the vicinity of a separation point $x = x_0$, we can relate (20) with (17) when we subtract $\kappa(x_0)(x-x_0)^2/2\epsilon$ from Y and identify time with arc length along the body and ξ with ψ. As described by the above equation, separation can now take two forms. First, there is the case of 'natural' separation where κ is so large at $x = x_0$ that the pressure at the base of the shock layer (or on the piston for (17)) falls to zero there. In this case we can re-scale $x-x_0$ with a suitable power of ϵ to retrieve an equation identical to (20) but with κ a constant and $\epsilon = 1$. Secondly, we can consider 'artificial' separation where κ is discontinuous at $x = x_0$. In this case the flow upstream of the separation point can be considered to be that past a wedge so that κ, $\tilde{F}$ and $\tilde{G}$ are effectively constants just upstream of $x = x_0$. Now a rescaling of $x-x_0$ with $\epsilon^{\frac{1}{2}}$ retrieves (19).

The most striking phenomenon we have encountered has been the global consequence of even small 'bluntness' effects in hypersonic flow. This fact was crucial to the early development of theories for hypersonic flow past thin or slender bodies and has formed the basis for our discussion of the piston problem for arbitrary power-law motion. The principal theoretical idea is that the blunting can be modelled as a localized drag, or energy release, and it is not surprising that energy dissipation mechanisms play such an important role in understanding inviscid hypersonic flows. What makes regularization with this kind of 'blunting' of mathematical interest is the fact that for a wide class of x_{p0}, piston motions with $x_p = ct^{\alpha}, \frac{2}{3} \geqslant \alpha > \frac{1}{2}$, always have the shock at $x_s = at^{\frac{1}{2}}$ to lowest order; only the parameter a is to be determined from x_{p0}. This situation may be contrasted with more conventional 'viscosity' regularizations.

Finally, we note that although our scenario for newtonian piston motion relies heavily on the concept of 'free layers' which separate from the piston under appropriate conditions, we have said nothing about the uniform validity of these layers for large times. Indeed, Freeman (1960) and Hornung (1969) have pointed out that perturbations to free layer solutions in the form of asymptotic expansions in ϵ show that the layers ultimately thicken and, according to Freeman (1962) and Hornung (1969) eventually evolve into blast waves.

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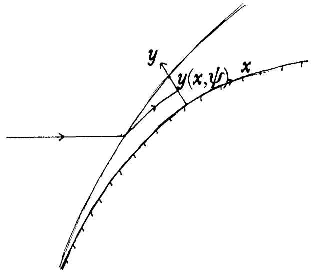

Figure 5.

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Appendix

When the gas flow is confined to a thin layer close to the body we choose a system of 'body-fitted' curvilinear coordinates, as shown in figure 5; x and y measure distances along and normal to the body respectively.

The equations of motion are then

$$\partial/\partial x(\rho u)+\partial/\partial y(h\rho v)=0, \tag{A 1}$$

$$u\,\partial u/\partial x+hv\,\partial u/\partial y+\kappa uv+(1/\rho)\,\partial p/\partial x=0, \tag{A 2}$$

$$u\,\partial v/\partial x+hv\,\partial v/\partial y-\kappa u^2+(h/\rho)\,\partial p/\partial y=0, \tag{A 3}$$

and

$$u\,\partial/\partial x(p/\rho^\gamma)+hv\,\partial/\partial y(p/\rho^\gamma)=0, \tag{A 4}$$

where u and v are the velocity components in the x and y directions, $\kappa(x)$ is the body curvature and $h=1+\kappa y$.

Equation (A 1) shows the existence of a stream function ψ such that $\partial\psi/\partial y=\rho u$ and $\partial\psi/\partial x=-h\rho v$; ψ is the steady flow counterpart of the 'particle-path' function ξ. The above equations may be simplified by taking x and ψ as independent variables, when the equations become

$$u\,\partial u/\partial x+\kappa uv+1/\rho(\partial p/\partial x-h\rho v\,\partial p/\partial\psi)=0,$$

$$\partial v/\partial x-\kappa u+h\,\partial p/\partial\psi=0,\quad u\,\partial/\partial x(p/\rho^\gamma)=0,\quad \partial y/\partial\psi=1/\rho u,$$

and

$$\partial y/\partial x=hv/u,$$

which can be viewed as the lagrangian counterpart of equations (3) for steady two-dimensional flow. The above procedure can also be carried out for axisymmetric flow with almost identical results.

The boundary condition on the body is $y=0$ on $\psi=0$ and the shock conditions become

$$\rho/\rho_0=(\gamma+1)/\{(\gamma-1)+2/M^2\sin^2(\phi+\delta)\},$$

$$p/p_0=\{2\gamma M^2\sin^2(\phi+\delta)-(\gamma-1)\}/(\gamma+1),$$

$$\frac{u}{U}=\cos\phi-\frac{2}{\gamma+1}\sin\delta\sin(\phi+\delta)\left[1-\left(\frac{1}{M\sin(\phi+\delta)}\right)^2\right],$$

$$\frac{v}{U}=-\sin\phi+\frac{2}{\gamma+1}\sin(\phi+\delta)\cos\delta\left[1-\left(\frac{1}{M\sin(\phi+\delta)}\right)^2\right],$$

on $y = y_s(x)$, where $\tan\delta = y_s'(x)$ and $\phi(x)$ is the body slope. In (12) we denote the distances of the body and the shock from the axis by $X_b(x)$ and $X(s)$ respectively, where $X_b = X_s + y_s\cos\phi$.

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